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AN OVERVIEW OF LUDWIG WITTGENSTEIN'S EARLY PHILOSOPHY: FROM *LETTERS* AND *NOTEBOOKS* TO THE *TRACTATUS LOGICO-PHILOSOPHICUS* AND A LITTLE BEYOND

"What a Copernicus or a Darwin really achieved was not the discovery of a true theory but of a fertile point of view."

(L. Wittgenstein, *Culture and Value*, 18)

Abstract: In his early philosophical work, Ludwig Wittgenstein developed a full range of ideas in metaphysics, philosophy of language and value, but also in the philosophy of logic, mathematics and natural science. The aim of the present paper is to discuss these ideas in relation to Wittgenstein's central metaphysical project, the "picture theory". My first claim is that "picture theory" grounded a semantic research in logic, mathematics and science that was maintained by Wittgenstein until the early 1930s. Secondly, I claim that Wittgenstein's solutions in semantics, especially after 1928, still make a pertinent philosophical project when evaluated from a contemporary perspective.

Keywords: Ludwig Wittgenstein, *Tractatus Logico-Philosophicus*, picture theory, philosophy of logic, foundations of mathematics, science, semantics.

Introduction

Throughout his entire life and career in philosophy, Ludwig Wittgenstein published but a single book. *Tractatus Logico-Philosophicus* appeared, with considerable struggle from the part of its author, at the end of the First World War in 1921. It was initially written in German, but it was translated in English immediately afterwards, in 1922, and was lavishly prefaced by the English philosopher Bertrand Russell. It is a small book – only fifty five pages long, but remarkably dense and difficult to grasp.

Every year, new readings claim to bring fresh insights into a philosophical work that seems to never age. Many succeed, but the peculiarity of Ludwig Wittgenstein's thought, which often seems to contradicting itself, gives the impression that a complete and satisfying interpretation will never be found. It is Wittgenstein's belief that one needs a special apprehension of what philosophy is in order overcome apparent contradictions and grasp the book, and that such an apprehension cannot be conveyed otherwise than metaphorically: philosophy is like a ladder that one has to climb and eventually drop because it is nonsense. Only someone who is willing to apprehend philosophy in this manner will succeed in giving the *Tractatus* a meaning.

Many early Wittgenstein scholars find themselves puzzled by this predicament. I find myself puzzled quite often. However, Ludwig Wittgenstein's famous claim in the *Tractatus* is just one way to look at his early philosophy. There are more ways. In his early philosophical *corpus* that comprises over ten years of written work starting with 1912, Ludwig Wittgenstein develops quite a range of philosophical ideas in metaphysics, philosophy of language and value, but also in philosophy of logic, foundations of mathematics and philosophy of science. These ideas have a life of their own in notebooks, letters and typed manuscripts remained unpublished – away from the systemic and austere constraints imposed in the *Tractatus Logico-Philosophicus* itself. Hence, no matter how important the contention about overcoming philosophical problems was to Wittgenstein in the advent of publishing the book, this is not the only perspective that one can set on Ludwig Wittgenstein's early philosophy.

Setting aside Wittgenstein's own belief, I aim here to explore and develop some of his early views in the philosophy of logic, mathematics and natural science. In the endeavor, I consider not only the oracular maxims found in the *Tractatus* on the topic, but also several unpublished documents from the period: Ludwig Wittgenstein's philosophical correspondence with Bertrand Russell from before, during and after the First World War; his *Notebooks 1914 –1916*, a collection of private notes that document the timely unfolding of his "picture theory" (*Bildtheorie*); and the *Prototractatus,* a provisional version of the published book that existed only in manuscript and became public long after Wittgenstein's death. In order to illustrate how these ideas had a life of their own in Wittgenstein's early philosophy, even years after the ladder metaphor became obsolete, three post-Tractarian sources are also relevant to consider: Wittgenstein's only published article "Some remarks on logical form" from 1929; the conversations with members of the Vienna Circle from 1929 and 1930 recorded by Friedrich Waissmann; and several philosophy lectures held in Cambridge between 1930 and 1931, recorded by some of Wittgenstein's own students. From 1929 to 1930 is the time when Wittgenstein began to reaffirm big themes of his previous work, while acknowledging that the framework in which the *Tractatus* was set was not necessarily the most fruitful one. In a private note from 1930, Wittgenstein writes: "Aside from the good & genuine, my book the Tractatus Log.-Phil. also contains kitsch, that is, passages with which I filled in the gaps and so-to-speak in my own style. How much of the book consists of such passages I don't know & it is difficult to fairly evaluate now."[1]

Starting from here, the first point I will make is a hermeneutic one. Ludwig Wittgenstein's goal in the *Tractatus* was to apply modern logic to metaphysics in order to show how human beings make sense of the world. To this end, he sought to figure out semantics – *i.e.* what happens, in principle, when one maintains that a description of the world is either true or false. This is what later became known as his "picture theory" (*Bildtheorie*). It is my claim that Wittgenstein's picture theory grounded a semantic research in logic, mathematics and natural science that was maintained with certain variations until the early 1930s, even though the Tractarian concept of picturing had been strictly abandoned.

The second point I will make is a philosophical one. To create links between logic, mathematics and natural science, in order to develop a knowledge of the world is an open philosophical endeavor still. Classical programs, like Bertrand Russell's semantic structuralism or Rudolf Carnap's syntactic formalism in the philosophy of mathematics and of natural science are in spotlight again and reconsidered. It is my claim that early Wittgenstein's research in semantic representations – developed especially in the late 1920s and early1930s – should also be on the list. After 1928, Ludwig Wittgenstein began to see semantic meaning as intricately interconnected with calculation, measurement and geometric projection. In this metamorphosis of "picture theory", the idea of true description was slowly enriched with something else – *i.e.* with various human activities of creating descriptions. In Wittgenstein's late 1920s philosophy, an idea was present that went clearly against both Russell and Carnap's programs. According to it, there is no proper semantics of logic, mathematics or science; but logic, mathematics and science provide the semantics (*i.e.*

the projection rules) for our representations in general. In other words, semantics is not "there", instead it is being made in order to make sense of the world.

1. The Hermeneutic Space

Here, I sketch a hermeneutic space for three important themes that occur in Ludwig Wittgenstein's early philosophy: logic, mathematics and natural science. The preferred interval for discussion is 1913 to 1930. It is my assumption that, between 1913 and 1930, Ludwig Wittgenstein's philosophical ideas did propagate a considerable set of infinitesimal echoes and variations, but they remained mostly in the same frame. Once this assumption is made, the selected interval can allegorically perform the function of a Cauchy sequence in mathematical calculus, so that the progression towards some limit is defined by setting up a closed value-space in which the limit itself is contained. It is hard, if not impossible, to pin-point the end of Ludwig Wittgenstein's early philosophy, but the existence of a closed space beyond which his views start to diverge drastically is somehow a guarantee that the limit exists.

The common element in this space is Ludwig Wittgenstein's "picture theory" (*Bildtheorie*) developed in the *Tractatus*, but quite older than the *Tractatus,* since its first elements already appear in the 1914 manuscript "Notes on Logic".[2] Equally remarkable, the "picture theory" does not disappear once the *Tractatus* is finished and published in 1921, but converts into a new semantics of human language in which representation spaces, which are basically mathematical spaces, replace the single Tractarian logical space. This transformation is more than obvious in "Some remarks on logical form" (1929), but it also propagates in Ludwig Wittgenstein's 1930 views of geometry and mechanics.

Therefore, in order to examine the development of Wittgenstein's ideas on logic, mathematics (including geometry) and science (especially physics) in his early philosophy, it is important to look at how the "picture theory" itself goes through difference stages in this closed interval that seems to contain its very limit.

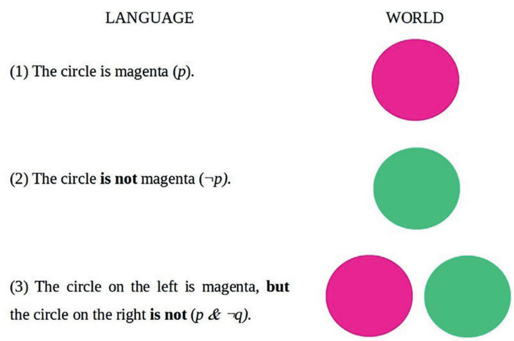
2. Elements of "Picture Theory" (*Bildtheorie*)

In broad lines, Ludwig Wittgenstein's "picture theory" (*Bildtheorie*) is a speculative theory about how human language works and acquires meaning – it is a philosophical answer to the question of what is going on when a proposition about anything in the world is true. For example, a sentence like "Snow is white" is true because it depicts something real about the world – *i.e.* the fact that snow, as we know it, is white and not black or green. Also, precisely because snow is white and not black or green, the sentence "Snow is green" is false, and also, because snow has the physical properties that it has, it is not only false, but actually meaningless to say that "Snow runs". The "picture theory" makes intuitive and straightforward complicated abstract concepts like truth, falsehood and meaningfulness of plain human language.

But what happens when we have to deal with logical propositions containing formal signs, such as logical operators '¬', '&', '∨' etc. or the sign for identity '='? Or when we have to make sense of mathematical equations? Or of physical laws, which are not simple propositions coding facts, but propositions coding various necessary correlations between facts? It is relevant to mention here that Ludwig Wittgenstein was looking, for instance, for a solution to the meaning of the identity sign '=' as early as 1913, but he was not very sure how to approach the issue. In a personal letter to Bertrand Russell, he was writing quite passionately: "Identity is the very Devil and *immensely important*; very much more so than I thought (…) I have all sorts of ideas for a solution of the problem but could not arrive yet at anything definite. However I don't lose courage and go on."[3]

This is actually where the substantial and counter-intuitive input of "picture theory" should enter and make a significant contribution with respect to meaning, truth and falsehood. It should make clear in which sense all those kinds of propositions (logical, mathematical) are meaningful, if at all. It should tell, eventually, something important about human knowledge (*i.e.* about the meaning of general scientific claims, like the principle of induction, which bear no picturing relation with anything in the world). Wittgenstein was aware of these challenges and he did embrace them. The letter from 1913 to Bertrand Russell was no accident in this respect – it showed his determination. Eventually, the Tractarian "picture theory", finished already by 1918, was a good theory of the specialized languages of logic, mathematics and physics, even though the account was not done in as straightforward fashion as in the case of factive human language, and not in a bulletproof manner either. We shall see why.

The main claim behind the Tractarian "picture theory" is that all meaningful propositions represent the world directly and all representation is done pictorially. In order to understand the claim, let us look at some simple examples. These are meant to showcase the pictorial nature of the relationship between a rudimentary language and a world made of just a few possible states.



Cases (1) and (2) depict a color world characterized by two possible elementary states: W_1= {magenta, cyan}, while case (3) depicts a color world characterized by four possible elementary states: W_2 = {$\text{magenta}_{\text{left}}$, $\text{cyan}_{\text{left}}$, $\text{magenta}_{\text{right}}$, $\text{cyan}_{\text{right}}$}.[4]

Now, case (1) is trivial because its pictorial truth-condition is evident. But what is the truth-condition for case (2)? The world cannot contain "negative facts", because that would generate an inflationary metaphysics and Wittgenstein simply rejected inflationary entities. So the solution must be different. What we know is that a negative proposition obtains whenever the affirmative proposition fails to obtain. But, in pictorial terms, what does it mean that the magenta circle fails to obtain? Wittgenstein's suggestion in his 1914 manuscript "Notes on logic" is almost a photographer's solution: a magenta image fails to obtain whenever the pixels are *inverted* in the picture. In other words, the negation of a proposition describing an elementary world-state can be expressed as the actual inversion of the elementary world-state. At this point, Wittgenstein called language "bipolar": each proposition that describes a world-state has both a positive and a negative (inverted, complementary) pole which is its negation '¬'.

Case (3) is a composite proposition (a logical conjunction with a negative conjunct), so its pictorial truth-condition is more complex than cases (1) and (2). In Wittgenstein's pictorial terms, the corresponding world-state should be two circles side by side, drawn in complementary colours. Here, conjunction is expressed pictorially as *concatenation*, while negation is expressed as before as inversion. In this ingenious way, Ludwig Wittgenstein explicates logical connectives through pictorial equivalents and so achieves a more substantial pictoriality of human language, that goes beyond intuitive factive accounts like "Snow is white" and dissolves logical operators like classic first-order operators (negation, conjunction etc.) into collages of pictures.[5]

To generalize, Wittgenstein's "picture theory" is built on five central assumptions:[6]

1) *Atomicity*: Individual words mirror (individual) objects. "Magenta" mirrors magenta, and "circle" mirrors circle.

2) *Categoricity* (similarity): Words are similar to objects*, i.e.* they bear the same formal properties and the same formal relations to each other. If magenta and cyan are two inverted colors, then the terms "magenta" and "cyan" are two complementary predicates, and *vice versa*.

3) *Compositionality*: Words form elementary propositions and elementary propositions form composite propositions, by means of logical operations. Case (3) is such a composite proposition.

4) *Pictoriality*: Elementary propositions picture elementary world-states.

5) *Truth-functionality*: Each composite proposition is a truth-function of elementary propositions (*TLP* 5).

Therefore, since every meaningful proposition is either elementary or composite and each composite proposition is a truth-function of elementary propositions, such that it preserves pictoriality, given that each logical operation is equivalent to a collage of pictures, then all meaningful propositions represent the world *directly* and all linguistic representation is done *pictorially.* The direct character of representation is obtained by simply assuming categoricity or similarity (Assumption 2) while the pictorial character has to be secured through a conservative extension of the elementary picturing relation (Assumption 4).

In order to be able to maintain this version of *Bildtheorie,* Wittgenstein has to show that, indeed, each logical operation is equivalent to a collage of pictures. Otherwise, the extension of pictoriality from the elementary to the composite cases would be non-conservative. On the other hand, if full pictoriality is shown, then the logical vocabulary of first-order logic, i.e. '¬',

'&', '∨', '⊃', could be dismissed as superfluous, and logical propositions would be shown to be very different from plain factive language in the sense of lacking descriptive content entirely.

This is where the strange proposition number 6 in *Tractatus Logico-Philosophicus* comes into full play: "The general form of truth-function is $[\bar{p}, \bar{\xi}, N(\bar{\xi})]$. This is the general form of a proposition." (*TLP* 6). Proposition 6 provides the notational device by means of which the pictoriality proof can be obtained unproblematically. Basically, proposition 6 says the following: given $\bar{p}$, the set of elementary propositions *p, q, r,…* in a language *L* and $\bar{\xi}$, the set of Sheffer truth-functions on $\bar{p}$ (*i.e.* negations and conjunctions of negations of *p, q, r,...*), then any composite proposition in *L* can be written as a n-*th* application of the Sheffer truth-function on $\bar{p}$. In plain words, a composite proposition is a picture if and only if it can be translated as a conjunction (concatenation) of elementary propositions (or pictures) taken in their negated (inverted) form.

The first step towards this result was achieved by the French logician Jean Nicod who, in 1917, proved that all first-order truth-functional language can be translated into Sheffer truth-functions by using a most minimal logical system.[7] Nicod's system had a single connective, the "Sheffer stroke" or '|', such that a sentence 'p|q' was assigned the meaning "not both p and q", and all connectives in first-order language were translated into Sheffer equivalents, like this:

i) '¬p' is equivalent to 'p|p';

ii) 'p&q' is equivalent to '(p|q)|(p|q)';

iii) 'p∨q' is equivalent to '(p|p)|(q|q)' and

iv) 'p⊃q' is equivalent to 'p|(q|q)'.

What Wittgenstein extracted from Nicod's proof was the certainty that any meaningful proposition, as far as it is either true or false, is equivalent to a Sheffer concatenation of only inverted elementary propositions. Nicod's proof, however, displeased him profoundly, because it was done by means of a procedure that he considered illegitimate. In the Preface of *Tractatus Logico-Philosophicus,* in 1919, Wittgenstein wrote emphatically "... in order to be able to draw a limit to thought, we should have to find both sides of the limit thinkable (*i.e.* we should have to be able to think what cannot be thought). It will therefore only be in language that the limit can be drawn, and what lies on the other side of the limit will simply be nonsense."[8] To prove that his minimal system was complete with respect to the expressiveness of first-order truth-functional language, Nicod had to introduce "from outside" in Wittgenstein's view both an arbitrary inference rule and an axiom for making derivations and substitutions,[9] which of course were meaningless. Neither the rule, nor the axiom were saying anything, but still they were held as "true". To use them in order to show the expressive completeness of Nicod's system meant for the young Wittgenstein to trespass the limit of meaningfulness itself.

So, in the 1917s, 1918s Wittgenstein was looking for another way to achieve expressive completeness with his *Bildtheorie* – this time "from within" language itself. He did not keep the Sheffer notation as such, but remained faithful to the important insight behind it. The young Wittgenstein was hoping that, by using only pictorial operators – like concatenation (conjunction) and inversion (negation) – he could reach the same result as Nicod did, but without having to work out any axioms and inference rules. He was determined to obtain expressive completeness by means of a generic procedure of constructing all truth-functions of first-order language as a series of sums. This result, however, was formally intractable. In fact, after 1928, he abandoned the idea entirely.

Yet, Wittgestein had discovered no later than 1916[10] that a constructive limit of any $[\bar{p}, \bar{\xi}, N(\bar{\xi})]$ sequence could be given as a tautology or contradiction of first-order truth-functional logic. Playing with his new T-F notation, he noticed that, after enough many iterations, any meaningful proposition can be made to converge towards a meaningless one. To the young Wittgenstein, this formal aspect of language could not have been a mere accident. On the contrary, to him such results were showing that the limit of sense could be drawn from within language itself – even though in a rather inexact manner.

I will illustrate this insight with a very simple $[\bar{p}, \bar{\xi}, N(\bar{\xi})]$ sequence:

1: $\neg p$

2: $\neg\neg p$

3: $\neg\neg p \,\&\neg\, p$

4: $\neg(\neg\neg p \,\&\, \neg p)$

5: $\neg\neg(\neg\neg p \,\&\, \neg p)$

Iteration 4 here expresses the law of the excluded middle which, in early Wittgenstein's philosophical idiom, is a meaningless tautology (*e.g.*, "It rains or it doesn't rain"). It is also nice to observe that iteration 5 is a continuation of the sequence, which remains in the limit, given that a meaningless contradiction is now obtained (*e.g.*, "It is not the case that it rains or it doesn't rain"). This neat result must have pleased Wittgenstein because it illustrates quite convincingly the mathematical idea of limit, which can be simply characterized like this: let *T=(S, τ)* be a discrete space with a discrete metric τ, called logical space. Let $N = [\bar{p}, \bar{\xi}, N(\bar{\xi})]$ be the general term of a truth-functional sequence in first-order language. N converges in T to a limit if and only if: $\exists k \in N$ and $\forall m \in N$, such that $m > k$ and $m\bar{\xi} = k\bar{\xi}$. Simply said, in a discrete space, a sequence N reaches a specific value of S and it just "remains there". That is the limit.

The fact that such a limit exists shows now beyond doubt, in young Wittgenstein's view, that logical connectives are indeed empty notational devices, which can be either dismissed or kept only in order to ease more fastidious pictorial representations. In this sense, all meaningful propositions represent the world directly and all linguistic representation is done pictorially, according to the *Tractatus*.

2. Early Views on Identity, Arithmetic and Mechanics

I will now discuss a few interesting consequences of this theoretical set-up for Wittgenstein's early semantics (before 1928) of identity, arithmetic and mechanics. Even if one accepts the "picture theory" as a proper account of meaningful plain language, certain things may still appear puzzling. For example, what are synthetic identity statements like "Cicero is Tullius" or "Hesperus is Phosophorus" really about – what facts do they picture? What do the apparently synthetic propositions of arithmetic like "$2 \times 2 = 4$" actually mean? What sense do the general laws of nature like Newton's laws of motion or the principle of least action convey, as long as Newtonian and analytical mechanics are absolutely equivalent as far as their empirical predictions go? In short, what can the "picture theory" tell about real human knowledge, *i.e.* about the *synthetic* propositions of mathematics and natural science?

Ludwig Wittgenstein's answers to these questions are quite sophisticated and suggest that his "picture theory" can indeed be a workable philosophical tool. I will start with arithmetic.

It is plainly obvious that "$2 \times 2 = 4$" is not a pictorial representation of any fact, so how are we to understand mathematical equalities? Wittgenstein's solution here is coherent with his previous considerations on linguistic meaning: mathematical equations are akin to tautologies in first-order truth-functional language; as there is a logical method to prove tautologies (*TLP* 6.1203), there is also a substitutional method[11] in arithmetic that allows to prove mathematical equations. Thus, Wittgenstein would say that a mathematical relation like "$2 \times 2 = 4$" holds if and only if the corresponding equation, which is framed in the language of a general theory of operations, can be proven. Again, he uses a fully constructive approach – which does not assume set theory or any formal theory alike.

His ingenious solution is to define natural numbers as generalized sums based on a abstract notion of operation: [*a, x, O'x*], where: (i) *a* is a 0 term, (ii) *x* is an arbitrary term, (iii) *O'x* is the form of the term that immediately follows *x*, obtained by applying operation *O* to *x* (*TLP* 5.2522). Then construction may proceed like this:

(*) $a, O'a, O'O'a, O'O'O'a, O'O'O'O'a\ldots$

which counts successive applications of the abstract operation *O*:

(**) $(O)^{0,}x, (O)^{0+1,}x, (O)^{0+1+1,}x, (O)^{0+1+1+1,}x\ldots (O)^{\sum_{n=0}^{\infty} n,} x$

Now, natural numbers are defined as the exponents of the series of sums in (**):

(***) $(O)^{0,}x, (O)^{1,}x, (O)^{2,}x, (O)^{3,}x, \ldots (O)^{n}x$ and so on.

Based on this formalization, a formal proof of "$2 \times 2 = 4$" might be sketched as follows:

$2 \times 2 = 4$ iff: $O^{2\times 2,}x = O^{4,}x$

(1) $(O^{\upsilon})^{\mu,}x = O^{\upsilon\times\mu,}x$, Def.

(2) $O^{2\times 2,}x = (O^{2})^{2,}x$, from (1) by substitution.

(3) $(O^{2})^{2,}x = (O^{2})^{0+1+1,}x = O^{2,}O^{2,}x$, by (**) and then by (***)

(4) $O^{2}O^{2,}x = (O'O')'(O'O')'x$ by (*)

(5) $(O'O')'(O'O')'x = (O'O'O'O')'x$, by associativity of addition

(6) $(O'O'O'O')'x = O^{0+1+1+1+1,}x = O^{4,}x$ by (**) and then by (***).[12]

This substitutional approach shows how the equality sign can be eliminated from arithmetic just like logical connectives can be eliminated from the truth-functional first-order language, once we proceed from a most general form of operation, to which both numbers and specific arithmetic operations can be reduced. There is no synthetic mathematical knowledge – but only clear-cut formal transformations in the general language of operations. The very idea of "mathematical equality" is only apparent in the substitutional character of mathematical proofs, quite in the same way in which the ideas of "negation" and "logical product" are apparent in the pictorial character of propositional language (as inversion and concatenation of elementary world-states). From this perspective, two important claims in *Tractatus Logico-Philosophicus* receive a clearer meaning at this point. "Mathematics is a method of logic." (*TLP* 6.234) and "The logic of the world which the propositions of logic show in tautologies, mathematics shows in equations." (*TLP* 6.22) The whole world scaffolding in built in the *Tractatus* on a most general notion of operation. $[\bar{p}, \bar{\xi}, N(\bar{\xi})]$ is in fact only a case of $[a, x, O'x]$.

Now, even synthetic identity statements like "Cicero is Tullius", which had puzzled young Wittgenstein for years, can be explained. If the equality sign '=' is a notational device in mathematics, what else can the identity sign from "Cicero = Tullius" be in logic? It is not clear yet how the identity sign ca be eliminated from logical vocabulary. It is, in fact, the categoricity[13] of *Bildtheorie* that led Wittgenstein towards finding a solution to the problem he had posed in 1913 as "Identity is the very devil!". In *Tractatus Logico-Philosophicus,* he had a way out. The proposition "Cicero = Tullius" is simply meant to say that the person called Cicero is the same as the person called Tullius. Now, if the mirroring relation between a name (word) and an object is *categorial* (*i.e.* it has exactly one form[14] up to isomorphism) then an identity copula '=' is redundant. In our case, the two names "Cicero"and "Tullius" have the same mirroring relation with a unique object of reference. Assuming categoricity, Wittgenstein simply claims that, if distinct names from a common category picked up the same object, then any proposition containing one name could be substituted with a proposition containing the other name without changing its meaning or truth-value.[15] Let's see the following substitutions:

(1) "Cicero wrote *De Natura Deorum*" is true.

(2) "Tullius wrote *De natura Deorum*" is also true.

(3) "Cicero was born in Arpinum" is true.

(4) "Tullius was born in Arpinum" is also true, and so on.

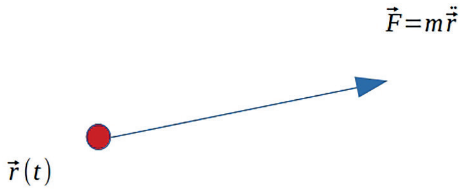
From them, it is obvious that the proposition "Cicero = Tullius" is simply superfluous or, better said, it indicates a misleading use of the very notion of identity. We could be under the wrong impression that the proposition expresses some kind of metaphysical fact, *i.e.* an identity of indiscernible objects as Gottfried Leibniz would have called it, but in fact it is not so. There are no metaphysical facts. What we have in "Cicero = Tullius" is just a substitutional equivalence between words. Of course, this equivalence rests on a significant number of conventions regarding, for instance, the manner in which specific names in a given category, like the category of male proper names, are chosen and assigned to objects. The use of conventions is ultimately what makes such an equivalence synthetic. Otherwise, an identity statement does not say anything real about the world –or not in the sense that Leibniz would have wanted it, anyway. For reasons of notational clarity, Wittgenstein even proposes in *Tractatus* the identity sign '=' be eliminated entirely from first-order language: "Identity of object I express by identity of sign, and not by using a sign for identity. Difference of objects I express by different signs." (*TLP* 5.53) The entire metaphysics of indiscernible objects is, thus, dismissed in *Tractatus Logico-Philosophicus* by means of a clear and perspicuous logical notation. Wittgenstein was, in fact, really convinced that a logical notation that does proper justice to the pictorial nature of human language can resolve metaphysical problems occurring not only in logic or in mathematics, but even in natural science.

The most general principles of nature (like the principle of causality or the principle of induction) are not, as one might expect, necessary synthetic propositions about the physical world, but only pseudo-propositions[16] (*TLP* 6.3). In classical physics, for example, one can easily replace *salva veritate* one entire physical theory for another: analytical mechanics – developed on the principle of least action,[17] could be applied instead of Newtonian mechanics – developed on the principle of causality,[18] in order to account for the same phenomena of physical motion. Two physical theories can be empirically equivalent, and this indicates the most general principles of nature do not say anything proper about the world. However, it is Wittgenstein's belief in *Tractatus* that physical laws play a constitutive role in human knowledge: "Newtonian mechanics, for example, imposes a unified form on the description of the world. Let us imagine a white surface with irregular black spots on it. We then say that whatever kind of picture these make, I can always approximate as closely as I wish to the description of it by covering the surface with a sufficiently fine square mesh, and then saying of every square whether it is black or white. In this way I shall have imposed a unified form on the description of the surface. The form is optional, since I could have achieved the same result by using a net with a triangular or hexagonal mesh. Possibly the use of a triangular mesh would have made the description simpler: that is to say, it might be that we could describe the surface more accurately with a coarse triangular mesh than with a fine square mesh (or conversely), so on. The different nets correspond to different systems for describing the world." (*TLP* 6.341)

Mechanics adopts different systems of description with different "meshes" in order to explain and predict events in the world. This can be illustrated with a very simple pictorial analogy. Consider a rudimentary world populated by a single mass point r acted upon by a force F. The motion function of r is $\vec{r}(t)$, and the resultant effect of the acting force F will be:

(1) $\vec{F} = m\vec{a} = m\ddot{\vec{r}}$.[19]

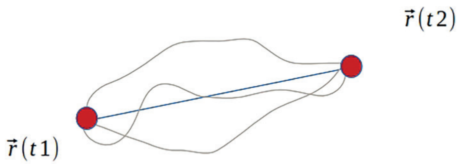
This intricate mathematical relation can be represented more intuitively once an abstract surface with a specific "mesh" is specified (the mesh characterizes the touching points between the abstract surface and a physical phenomenon). In our rudimentary world, the mechanical mesh consists of a single point in accelerated displacement (the mass point mirroring the physical body in motion).



In analytical mechanics, however, an entirely different mesh is at work: for each possible path (or displacement $\vec{r}$) of the mass point r, a quantity called *action* is defined such that it expresses a minimal variation in energy between the initial and final times, t_1 and t_2, of r's motion:

(2) $S(\vec{r}) = \int_{t_2}^{t_1} (KE\vec{r} - PE\vec{r})\, \mathrm{d}t$.

The corresponding surface looks intuitively like this (with two touching points that mirror the initial and final motion state of the physical system).



Now, from the worlds' point of view, (1) and (2) are equivalent descriptions, even though they picture the physical world differently. This case resembles "Cicero = Tullius", in the sense that the only significant difference between theories (1) and (2) is ultimately a conventional one: equation (2), which is an integral, corresponds to a coarser mesh applied to the physical world than (1). Equation (1), which is a second-order differential equation, generates a finer-grained mesh because it accounts for local physical displacement, not just for a global variation in energy. On the other hand, theory (2) is *simpler* than (1) because theory (2) fully ignores instantaneous behavior which can be irrelevant or sheer intractable in case of too many interactions. The most important virtue of theory (2) is, in fact, an epistemic one: it does not say more about reality than the other one does, it only says it more parsimoniously.

But epistemic values, like simplicity or even clarity, are important in science. The young Wittgenstein was strongly influenced in this belief by a treatise in classical mechanics he had studied during his engineer years in Manchester: Heinrich Hertz's *Principles of Mechanics Presented in a New Form* (1894).[20] Hertz had suggested that even a scientific theory is facing some "choices" – because any scientific theory has, embedded deep in its equations, a conceptual form. In our particular case, the form of Newtonian mechanics is deterministic (as it is a theory of forces) while analytical mechanics is actually not (as it relies exclusively on a calculus of variations). In this respect, it makes no sense to believe in a "deterministic" or in a "variational" world. These are just epistemic choices. But then, again, the question resurfaces: how is the progress of science possible? In what sense is scientific knowledge ultimately synthetic?

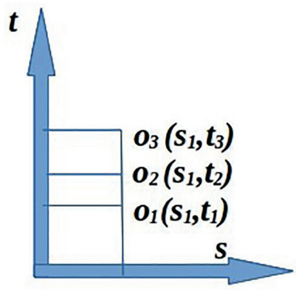
Inspired by Hertz, Wittgenstein viewed scientific progress as primarily a matter of conceptual transformations, which are transformations in form (quite like religious conversions, Wittgenstein believed: nothing in the world changes; change affects only the subject who experiences the world, the form of the experience, not its content; change is transcendental). The same holds with science as well as with philosophy: wherever conceptual forms are present, also conceptual problems arise, and conceptual transformations are expected. In this respect, it is not that Copernicus, Darwin or Einstein told us something "truer" about the world or that they discovered absolutely new facts when they introduced their theories of planets, life and universe; what made their science better was their ability to master profound conceptual transformations, so that more satisfying accounts of facts were eventually attained. These are, according to Wittgenstein, the sort of conversions that are really necessary for the real advancement of human knowledge – *i.e.* new conceptual forms that trigger better representations of the world. For example, in Newtonian mechanics, a bizarre thing happens when the third law of motion is considered. Let's think for a moment of an object attached to a rod that is rotated such that the object moves on a circular trajectory around some axis. The object is kept on its curved path by the centripetal force that "pulls" the object towards the rotation axis and, according to Newton's third law, by a second force equal in magnitude and opposed in orientation to the centripetal force, called centrifugal force, which prevents the object from actually "falling" into the axis. This representation is neat and seems to explain satisfactorily the observed phenomenon. However, centrifugal force is a strange notion. It does not do anything in the explanation. What counterbalances centripetal motion in our example is the object's own inertia to remain on a straight path, and not a second force. It is both Hertz's and Wittgenstein's belief that this conceptual difficulty was generated by the form of Newtonian theory, that postulated counteracting forces everywhere in nature. So, if a better explanation of circular motion is desired, then the classical theory of mechanics needs to be brought to a coherent form. This can be achieved, in Wittgenstein's opinion, through a proper "meshing" of theory onto the world, without idle concepts – that is, through an adequate construction of *the general form of mechanical theory*, and not by discovering new facts (*e.g.* centrifugal behaviors).

3. Later Stages of Picture Theory (1928-1930)

In 1922, Ludwig Wittgenstein decided to retire from professional philosophy.[21] He was firmly convinced that everything was settled in the *Tractatus*. In 1928, however, after few uncertain years, Wittgenstein met in Vienna Moritz Schlick, who was the leader of a small philosophical group called the Vienna Circle, and under his influence slowly began to reassess claims from his earlier work. He even befriended Schlick in whom he discovered, quite unexpectedly, a close and kindred spirit.

During the years 1928 to 1930, Ludwig Wittgenstein's picture theory (*Bildtheorie*) takes a fast and surprising evolution. Some of the previous core assumptions about representation are dropped without much ado: atomicity, categoricity, even truth-functionality. Slowly, Wittgenstein starts to imagine a different way of making sense of the world, still underlined by a general idea of representation (*Abbildung*), but not it the strict sense of before. His old theory of logic that was scaffolding his very concept of "world" begins to crumble after 1928, giving way to a diversity of systems of representation, intuitive human calculi and applied geometry. The world looks, all of a sudden, a lot more complex, more patchy and under the constraints of actual human abilities to make sense of it. Even the idea of a "true" representation gets replaced with that of a perspicuous representation, *i.e.* one that possesses the right mathematical multiplicity in particular a system of representation but is not the exact mirror of what it represents. There are "so many logical forms", Ludwig Wittgenstein writes in his short paper from 1929 entitled "Some remarks on logical form", and his revisionary work starts to illustrate this idea more and more convincingly.

One can imagine a very simple example, a sentence like: "There were three knocks on the door". Now, what does it mean for this sentence to be true? Should three knocks exist in the world? Does the sentence really mirror knocks? At this point it becomes obvious to Wittgenstein that it is misleading to refer to knocks on door as existing objects that our words mirror by similarity. Something else is the case here. Knocks on door may be something real, but only in a very particular system of representation – for instance, in a system of sound and time scales. Sound and time scales, on the other hand, are human constructs which involve intricate projective calculi: *i.e.* defining a sound and a time unit, counting such units (by means of, let's say, the general operation *O*), composing them into n-dimensional scales by forming Cartesian products thereof. A simple sentence like "There were three knocks" is actually a quite complex representation like this one:



This is just a schematic example of how the assumption of categoricity (according to which words mirror objects by a relation of similarity) simply fails. Along with it, however, other fundamental assumptions of "picture theory" fall as well. A straightforward case is that of truth-functionality, *i.e.* the assumption that each composite sentence is a truth-function of elementary sentences to which the truth-operation *N* is applied. We want to say, for example, that "It is not the case that there were three knocks". The latter is a composite sentence obtained by applying sentential negation to our supposedly elementary sentence: "There were three knocks". But what state of affairs does the negated sentence depict? As we may assume from the *Tractarian* concept of direct representation, it should depict the inverted or complementary state of three knocks. However, in this case, an infinite number of alternative states is possible: from no knock to several knocks, provided enough time. So there is no inverted world-state for three knocks.

As Jaakko Hintikka pointed out with respect to the development of Ludwig Wittgenstein's picture theory in the late 1920s: "Complex propositions have to be projected on reality, *i.e.* connected with atomic propositions, in some other way than truth-function theory. And it is clear from what Wittgenstein says that his first candidates for this role are the calculi that operate with numbers".[22] Projective calculi are more effective in making explicit the meaningful connection between human language and reality than the coarser grained logic of tautologies, of the kind "$p \vee \neg p$". In many representation cases, the excluded middle simply does not hold because, in between the logical extremes, an entire variety of intermediary world-states is possible. Applied mathematics and applied geometry can definitely account for those, a lot more perspicuously than logic does. Wittgenstein's revised concept of representation (*Abbildung*) from 1929 suggests that calculation and projection are, in fact, deeply embedded in the manner in which human beings make sense of the word – even in most elementary descriptions. This is what *shows* itself in human language and not an immutable logical structure of the world.

Another aspect of representation that now falls short for Wittgenstein is the very idea of atomicity, which he had most probably gotten from Bertrand Russell as early as his first years in Cambridge. Atomicity is, basically, the claim that individual words mirror individual objects: "table" mirrors table, "chair" mirrors chair, "dog" mirrors dog and so on. Let us suppose now that we want to talk about a very simple measurement, like: "This wooden plank is 2 meters long". What does "2 meters" here mean? What individual objects are thereby mirrored?

It is quite salient that no individual objects called "meters" exist. When one counts "2 meters" it is not as if two individual objects are concatenated one next to the other, as the atomicity assumption requires. If that were to be the case, then the sentence "This wooded plank is 2 meters long" should be written as "This wooden plank is 1 meter long *and* this wooden plank is 1 meter long". It should be similar to: if "The cat and the dog are sitting on the mat", then it is the case that "The cat is sitting on the mat" and

"The dog is sitting on the mat", since cat and dog are individual objects for which independent predication is required. Yet, this predicative approach does not work in the wooden plank case. Here, by contracting redundant conjuncts, we simply obtain the sentence "This wooden plank is 1 meter long" which obviously is not the same as "This wooden plank is 2 meters long". The example illustrates plainly that, in measurement representations, units are not just concatenated, but they form measurement scales for which a system of representation (*e.g.* a ruler) and an internal relation (*e.g.* addition) are stipulated. Someone has to perform such stipulations in order to get a sense of many things: from lengths, colors and sounds that make the universal human phenomenology, to intricate metric determinations of physical space that make the advanced science. This means that not even physical space is an object in itself, as some would still believe in the footsteps of Newtonian metaphysics. Numbers, scales and metrics are deeply entrenched with language, as part of various systems and calculi that serve to represent the world.

Ludwig Wittgenstein's late "picture theory" (1928-1930) is, in fact, a complex theory worth some particular attention in its own right. This is a point I find very important to underline. However, I will not be able to insist too much on it here. I mention only three salient aspects: late "picture theory" (a) preserves a representation (*Abbildung*) relation between language and world; (2) is characterized by a mathematical scaffolding (instead of a logical one) as calculus and projection are inherent parts of it and (3) is practical rather than transcendental; for instance, there is not an *a priori* notion of physical space that frames human experience, but rather it is human experience that frames a certain notion of physical space.

4. Semantic Research in 1930

In this final part, I shall illustrate the development of Ludwig Wittgenstein's view of geometry in relation to mechanics, development that was motivated by his partial revision of "picture theory" in the late 1920s. As I was beginning to suggest at the end of the previous section, the revised "picture theory" infuses a new and more nuanced understanding of what scientific knowledge ultimately is. Once one abandons the idea that theories are transcendental, the metaphor of a meshed grid imposed onto the world by a transcendental subject, touching reality only into its distant nodes, is also starting to shimmer. Scientific theories of physical space are constructive human endeavors; each conceptible spatial mesh is perspicuously sewed onto the world with calculations and geometrical projections that make sense for beings who experience space, in a way or another.

In 1929, in a conversation with Moritz Schlick and Friedrich Waissmann in Vienna, Wittgenstein introduced an elaborate concept of geometry as syntax of physical space: "Einstein says that geometry is concerned with possible positions of rigid bodies. If I actually describe the positions of rigid bodies by means of language, then it is only the syntax of this language that can correspond to possible positions."[23] In this view, geometry provides *constitutive rules* for the expression of spatial properties and relations; that is, geometry is understood as a grammar of physical language. Geometry is used to describe physical space, but also to constrain what spatial rapports are possible or impossible between physical objects. In other words, mechanics, for example, is always constrained by the kind of geometry one has – since, depending on the particular geometry, certain trajectories in space are conceptible or not. In this sense, both Wittgenstein and Einstein were quite fond of applied geometry, just like Hertz had been several decades before.[24] Geometric conventions make sense only in as much as they are involved with real objects of experience.[25]

The suggestion, which Wittgenstein did not explicate any further, can be illustrated with a very nice and intuitive example. One can draw the following table:[26]

Spatial relation	**Kind of geometry (in which the spatial relation is expressible)**
(1) The pencil is in the box. (box is closed)	Topology
(2) The pencil is in the box. (box is open) Mary is sitting between Jose and Maria.	Affine geometry
(3) The post office is over the hill.	Projective geometry

This tables offers an intuitive glimpse into the geometry (or geometrical "syntax") that makes the semantics of various spatial prepositions in English. It works with the general simple idea of spatial invariance. Intuitively, in the case of: "The pencil is in the box" and the box is closed, the spatial relation "_in_" is invariant under continuous deformations of the box, so one could say that the use of preposition "_in_" from (1) is underlined by a notion of topological invariance. Now, if the box is open, one can easily imagine deformations of the object – *i.e.* the five-wall container plus the pencil – that do not preserve the "_in_" spatial relation; for instance, one bends the five disconnected walls in opposite directions. However, in the case of: "The pencil is in the box" and the box is open, the "_in_" relation can be preserved once one sees, if the example is simplified to a two-dimensional representation, the pencil line as being in the same plane as the three-line container. An invariance under affine transformations – which preserve co-planarity – is thus obtained. The other example: "Point *x* is between point y and point *z*" is meant to suggest that the "_between_and_" preposition is also invariant under affine transformations, since affine geometry preserves also collinearity. The last case is even a simpler one: a figure (*i.e.* a post office) is projected in a different plane while preserving the relative distances between its points. So "_over_" expresses an invariance under translation, which is characteristic of projective geometry.

Carl Hempel later called this kind of view a semantic interpretation of geometry; it was definitely influenced by Albert Einstein's "Geometrie und Erfahrung" (1921) and it had a lot to do with Wittgenstein's new notion of representation as a human activity based in human experience. Einstein had explicitly claimed that, in mechanics, by studying the movement of rigid bodies with geometric tools, one can formulate rigorous statements about real bodies' motion capabilities – because geometry is concerned with real objects of experience and their constrains. For Wittgenstein, things reached a little further, since even the logic of plain spatial language could be shown to be entrenched with rudimentary geometrical representation.

This idea was discussed at length in the Vienna Circle during the 1930s, and it is really remarkable and also a little puzzling that Wittgenstein called geometry "a syntax" of physical space. Both Schlick's and the Vienna Circle's view of geometry, including Carnap's, was actually more formal than Wittgenstein's. Carnap's view was, in fact, a properly syntactic one: it was Carnap's belief that primitive notions of geometry like "point", "line", "plane", together with the axioms and postulates should be understood independently of any intuitive manipulation of words, shapes and physical objects in bi-dimensional or three-dimensional spaces. In Carnap's sense, geometric notions have no empirical bearing, *i.e.* they are entirely formal. Wittgenstein, on the other hand, who was pairing with Einstein in this heavy polemic, maintained that geometry should be conceived of as embedded in human experience and, thus, in human language. So, geometry was not purely formal or syntactic. Against Carnap and even Schlick, Wittgenstein took a system of geometry, like Euclidean geometry, to be a system of rules for *applying* specific geometric notions such as "point", "line" or "plane" in different empirical contexts, including mechanics. From this particular perspective, that Wittgenstein was inclined to endorse towards the beginning of the 1930s, arithmetic and geometry were both formalisms, but not just formalisms. Arithmetic and geometry did not make sense just as such, by stipulation; what gave these systems meaning was their straightforward application from which they could not be separated.

Ludwig Wittgenstein started in Bertrand Russell's philosophical tradition from Cambridge in the early 1910s. Simplifying, his first philosophy up to the publication of *Tractatus Logico-Philosophicus* in 1921 was a mix of both embraces and challenges to Russell's ideas, some of which were more or less direct. One good example is the atomistic assumption of *Bildtheorie*, which both resembled and differed from Russell's. But the years starting from 1929 leave the impression of a deeper philosophical awakening for Wittgenstein. Russell's logical analysis of language is definitely left behind; and the proposal to characterize semantic aspects of human language based on mathematical and geometrical forms of linguistic representation is, indeed, entirely new. It is also productive because the notion of invariance may be just as worth exploring in semantics, as it is in mechanics. Yet, this discussion will have to be postponed for another time.

NOTES

1 L.Wittgenstein, *Public and Private Occasions,* p. 39.

2 See also M. Potter, *Wittgenstein's Notes on Logic,* p. 224-226.

3 L. Wittgenstein, *Cambridge Letters,* p. 42.

4 For the general case, the number of possible elementary states is 2^n (*TLP* 4.27, 4.28).

5 Wittgenstein calls it in the *Tractatus* a "fundamental thought" *(Grundgedanke*): "My fundamental thought is that the logical constants do not represent. That the logic of the facts cannot be represented." (*TLP* 4.0312)

6 See J. Hintikka, "An Anatomy of Wittgenstein's Picture Theory", pp. 21-54.

7 J. G. Nicod, "A Reduction in the Number of Primitive Propositions of Logic", pp. 32-34.

8 L. Wittgenstein, *Tractatus Logico-Philosophicus*, pp. 3-4.

9 The inference rule is a *modus pones*: from $A|(B|\Gamma)$ and A, Γ can be inferred. The axiom expresses the condition of *well-formation* for any Sheffer formula:

$$(A|((B|\Gamma))|(\Delta|(\Delta|\Delta))|((E|B)|((A|E)|(A|E)).$$

10 See M. Potter, "Wittgenstein's Pre-*Tractatus* Manuscripts: A New Appraisal", pp. 13-33.

11 "If two expressions are combined by means of the sign of equality, that means that they can be substituted for one another. But it must be manifest in the two expressions themselves whether this is the case or not. When two expressions can be substituted for one another, that characterizes their logical form." (*TLP* 6.23)

12 For complete proofs, see P. Frascolla, "The ' Tractatus' *S*ystem of Arithmetic", pp. 353-378.

13 Categoricity is a notion that Ludwig Wittgenstein knew from the foundations of mathematics, where it was applied for the first time by Richard Dedekind to prove that the axioms of arithmetic have exactly one model up to isomorphism, at the end of the 19th century. However, such uses of formal methods in Wittgenstein's early philosophy should not be taken in their strict mathematical sense, but in a rather speculative manner. Brian McGuinness calls them "snippets of method" (*Approaches to Wittgenstein,* p.165) because Wittgenstein never developed them into full formal applications.

14 See J. Hintikka,"An Anatomy of Wittgenstein's Picture Theory", p. 22: "Each name has the same logical form (logical and categorial type) as the object it represents".

15 "Just as we are quite unable to imagine spatial objects outside space or temporal objects outside time, so too there is no object we can imagine excluded from the possibility of combining with others" (*TLP* 2.0121)

16 For a more comprehensive exposition of this idea, see B. McGuinness, *Approaches to Wittgenstein*, pp. 117-118.

17 In analytical mechanics, a physical system of *n* mass points is characterized by a total quantity of motion which is expressed as a global least difference between the potential and kinetic energies of the system.

18 In Newtonian mechanics, motion is the response of a mass point to various forces, when considered in relation to other mass points. The action of forces is local and underlined by the idea of causality. Motion is obtained by changing a body's state through interaction with other bodies: through pushing, pulling, dropping etc.

19 The motion effect is equal to the acceleration of the mass point along its trajectory of motion. This can be written down as the second derivative of the mass point's displacement through time.

20 "The admiration that Wittgenstein conceived for Hertz in his youth was something he never lost. Later in life we find him entering reservations about almost everyone else – even about Frege – but right up to the end of his years he continued to quote Hertz with approval and agreement." (A. Janik and S. Toulmin, *Wittgenstein's Vienna,* p. 275)

21 See especially R. Monk, *Ludwig Wittgenstein: The Duty of Genius*.

22 J. Hintikka, "Die Wende der Philosophie: Wittgenstein's New Logic of 1928", p. 85.

23 L. Wittgenstein, *Wittgenstein and the Vienna Circle*, p. 28.

24 The physicist Heinrich Hertz was on the same page when he formulated his "geometry of systems of points": "… all our statements represent possible experiences; they could be confirmed by direct experiments, by measurements made with models. Thus we need not fear the objection that in building up a science dependent on experience, we have gone outside the world of experience." (H. Hertz, *The Principles of mechanics presented in a new form,* p. iii)

25 "...it is certain that mathematics generally, and geometry in particular, owes its existence to the need which was felt of learning something about the behavior of real objects. It is clear that the system of concepts of axiomatic geometry alone cannot make any assertions as to the behavior of real objects of this kind, which we will call practically-rigid bodies. To be able to make such assertions, geometry must be stripped of its merely logical-formal character by the coordination of real objects of experience with the empty conceptual schemata of axiomatic geometry. To accomplish this, we need only add the proposition: solid bodies are related, with respect to their possible dispositions, as are bodies in Euclidean geometry of three dimensions. Then the propositions of Euclid contain affirmations as to the behavior of practically-rigid bodies. We will call this completed geometry 'practical geometry'." (A. Einstein, "Geometrie und Erfahrung", the English translation: http://pascal.iseg.utl.pt/~ncrato/Math/Einstein.htm).

26 The source of these examples is P. Suppes, *Representation and Invariance* of *Scientific Structures*, p. 106.

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